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Averaging Sets on the Unit Circle

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For every normalized measure σ on the unit circle $\mathbf{T}$ let $t_\sigma(n)$ be the maximal integer t such that the quadrature formula of Chebyshev type

$$\int p(x, y)\, d\sigma = \frac{1}{n} \sum_{k=1}^{n} p(x_k, y_k)$$

holds for some subset $\{(x_1, y_1), \ldots (x_n, y_n)\}$ of $\mathbf{T}$ and for all polynomials $p(x, y)$ of $\deg p \leq t$. If ω is the Lebesgue measure then $t_\omega(n) = n - 1$. Moreover, $t_\sigma(n) \leq n - 1$ for every σ. Under the Kolmogorov–Szegö condition on σ we prove that $\sigma = \omega$ if $t_\sigma(n) = n - 1$ for a subsequence of $n = 1, 2, 3, \ldots$.

Let σ be a normalized positive regular Borel measure on the unit circle $\mathbf{T} = \{z : |z| = 1\}$ in the complex plane $\mathbf{C}$. Given a positive integer t, a subset $A = \{z_k : x_k + iy_k\}_1^n \subset \mathbf{T}$ is called a *σ-averaging set of degree t* if

$$\int_{\mathbf{T}} p(x, y)\, d\sigma = \frac{1}{n} \sum_{k=1}^{n} p(x_k, y_k) \tag{1}$$

for all polynomials $p(x, y)$ in the real variables x, y ($z = x + iy$), $\deg p \leq t$. Formula (1) is a Chebyshev type quadrature formula of degree t with the nodes z_k, $1 \leq k \leq n$.

A general result [6] provides the existence of σ-averaging sets for fixed t and all big n, $n \geq n(t)$. For the Lebesgue measure ω the averaging sets of degree t are called *t-designs*. In particular, every regular n-gon is a $(n - 1)$-design [1].

The condition (1) can be reduced to an equivalent complex form.

LEMMA 1. *A set $A = \{z_k\}_1^n$ is a σ-averaging set of degree t iff*

$$\int z^s\, d\sigma = \frac{1}{n} \sum_{k=1}^{n} z_k^s, \qquad 1 \leq s \leq t \tag{2}$$

where $\int \equiv \int_{\mathbf{T}}$ for short.

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Proof. Equation (1) with $\deg p \leq t \Rightarrow$ (2) with $s \leq t$ since $z^s = (x + iy)^s$ is a polynomial in x, y and its degree is s. Conversely, let (2) with $1 \leq s \leq t$ be given. Then the same equalities are valid for $|s| \leq t$ since σ is real and normalized. It remains to note that $p(x,y) = q(z, \bar{z})$, where q is a polynomial of the same degree, and it is a linear combination of z^s for $|z| = 1$, $|s| \leq \deg q$. ∎

COROLLARY 2. *If $A = \{z_k\}_1^n$ is a σ-averaging set of degree t then*

$$\int z^s \, d\sigma = \frac{1}{n} \sum_{k=1}^{n} z_k^s, \qquad |s| \leq t.$$

Remark 3. In the Lebesgue case, $\sigma = \omega$, all these integrals with $s \neq 0$ are zero, so A is a t-design iff

$$\sum_{k=1}^{n} z_k^s = 0, \qquad 1 \leq s \leq t. \tag{3}$$

It yields for $t = n - 1$ that A is a $(n - 1)$-design iff A is a regular n-gon (cf. [1]).

Let us denote by $t_\sigma(n)$ the maximal degree of σ-averaging sets of cardinality n. In particular,

$$t_\omega(n) = n - 1$$

since the system (3) with $t = n$ has only the trivial solution $z_1 = 0, \ldots, z_n = 0$. On the other hand, well-known arguments show that if $\operatorname{card}(\operatorname{supp} \sigma) > n$ then

$$t_\sigma(n) \leq n - 1. \tag{4}$$

Indeed, let $A = \{z_k\}_1^n$ and let $\rho = \rho_A$ be the measure concentrated on the set A and uniformly distributed on it, $\rho(\{z_k\}) = 1/n$ for $1 \leq k \leq n$. If now

$$\int z^s \, d\sigma = \frac{1}{n} \sum_{k=1}^{n} z_k^s = \int z^s \, d\rho, \qquad |s| \leq n$$

then

$$\int |P(z)|^2 \, d\sigma = \int |P(z)|^2 \, d\rho$$

for all polynomials P of degree $\leq n$. For

$$P(z) = \prod_{k=1}^{n} (z - z_k) \tag{5}$$

we obtain

$$\int |P(z)|^2 \, d\sigma = 0$$

which contradicts our assumption on σ.

From now on we assume that the support of σ is infinite, therefore (4) is valid. Our main result is the following

THEOREM 4. *Let the Kolmogorov–Szegö condition be fulfilled, i.e.,*

$$\int \ln \sigma' d\omega > -\infty \tag{6}$$

where σ' is the derivative of the absolutely continuous part of the measure σ with respect to ω. If for arbitrarily large n there exist σ-averaging sets $A_n = \{z_{n1}, \dots, z_{nn}\}$ of degree $n - 1$ then the measure σ is Lebesgue, i.e., $\sigma = \omega$.

In other words,

$$t_\sigma(n) < n - 1$$

if the measure σ under the condition (6) is non-Lebesgue and n is big enough. In this sense the only Lebesgue measure is extremal with respect to degrees of averaging sets.

We obtain Theorem 4 as an immediate consequence of two lemmas proved below.

LEMMA 5. *Suppose that there exists a subsequence S of sets $A_n = \{z_{nk}\}_{k=1}^{n} \subset \mathbf{T}$ such that*

$$\lim_{n \in S} \left(\int z^s \, d\sigma - \frac{1}{n} \sum_{k=1}^{n} z_{nk}^s \right) = 0, \qquad s = 1, 2, 3, \dots . \tag{7}$$

Then $\sigma = \omega$ iff

$$\lim_{n \in S} \left| \prod_{k=1}^{n} (z - z_{nk}) \right|^{1/n} = 1 \tag{8}$$

uniformly on every disk $|z| \leq r < 1$.

Note that Lemma 5 does not require the Kolmogorov–Szegö condition for σ.

Proof. Let us define the generating function

$$G_\sigma(w) = \sum_{s=0}^{\infty} w^s \int z^{-s-1}\, d\sigma = \int \frac{d\sigma}{z - w}$$

in the open disk $|w| < 1$. Setting $\rho_n = \rho_{A_n}$ we also consider the functions

$$G_n(w) \equiv G_{\rho_n}(w) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{z_{nk} - w} = -\frac{1}{n} \frac{D_n'(w)}{D_n(w)},$$

where

$$D_n(w) = \prod_{k=1}^{n} (w - z_{nk}).$$

Obviously,

$$G_\sigma(w) + \frac{1}{n} \frac{D_n'(w)}{D_n(w)} = G_\sigma(w) - G_n(w) = \sum_{s=0}^{\infty} \varepsilon_{n,s} w^s, \tag{9}$$

where

$$\varepsilon_{n,s} = \int z^{-s-1}\, d\sigma - \int z^{-s-1}\, d\rho_n.$$

Because of (7) $\lim_n \varepsilon_{n,s} = 0$ for $s = 0, 1, 2, \dots$. In addition, $|\varepsilon_{n,s}| \le 2$. If $\eta > 0$ and $r < 1$ are fixed, we can choose an integer m such that $2r^m(1 - r)^{-1} < \frac{1}{2}\eta$. There exists an integer N such that $\sum_{s=0}^{m-1} |\varepsilon_{n,s}| < \frac{1}{2}\eta$ for $n > N$. Therefore,

$$\left| \sum_{s=0}^{\infty} \varepsilon_{n,s} w^s \right| < \eta$$

for $n > N$ if $|w| \le r$. This means that the right side of (9) tends to zero as $n \to \infty$ uniformly in the disk $|w| \le r$. Thus,

$$\lim_{n \in S} \sup_{|w| \le r} \left| G_\sigma(w) + \frac{1}{n} \frac{D_n'(w)}{D_n(w)} \right| = 0. \tag{10}$$

Now let us note that $\sigma = \omega$ iff the Fourier coefficients $\gamma_s = \int z^s\, d\sigma$ are zero except γ_0. Therefore, $\sigma = \omega$ iff $G_\sigma(w) \equiv 0$. Further, it is equivalent to

$$\lim_{n\in S}\sup_{|w|\le r}\left|\frac{1}{n}\frac{D_n'(w)}{D_n(w)}\right| = 0 \tag{11}$$

because of (10).

At this point we can use the formula

$$\sqrt[n]{|f(w)|} = \exp\left(\operatorname{Re}\frac{1}{n}\int_0^w \frac{f'(z)}{f(z)}\,dz\right)$$

which is valid for every analytic function f in the unit disk having no zeros in it and such that $|f(0)| = 1$. For this reason (11) $\Rightarrow$ (8) uniformly on every disk $|z| \le r < 1$ and, conversely, (8) implies that

$$\lim_{n\in S}\operatorname{Re}\frac{1}{n}\int_0^w \frac{D_n'(z)}{D_n(z)}\,dz = 0$$

uniformly as well. The classical Schwarz formula yields

$$\lim_{n\in S}\sup_{|w|\le r}\left|\frac{1}{n}\int_0^w \frac{D_n'(z)}{D_n(z)}\,dz\right| = 0$$

and then (11) follows by the Cauchy formula for the derivative. ∎

The next lemma deals with general quadrature formulas, not only of Chebyshev type. Every such formula for the given measure σ has a form

$$\int z^s\,d\sigma = \int z^s\,d\tau, \qquad 0 \le s \le d, \tag{12}$$

where τ is a normalized positive measure with a finite support on the unit circle. Similarly (4) we have $d \le n_\tau - 1$, where $n_\tau = \operatorname{card}(\operatorname{supp}\tau)$. If d is the maximal possible for a given τ, $d = d_\tau$, then it is called the *degree of* τ (or the quadrature formula (12)).

For every n there exists a quadrature formula (12) with a measure τ such that $n_\tau = n$, $d_\tau = n - 1$. Basically, this follows from some general results concerning the trigonometric moment problem [5, Chap. 4] but one can establish this more directly [4, Sect. 7].

Now let us set

$$D_\tau(z) = \prod_{k=1}^{n}(z - z_k),$$

where $\{z_1, \dots, z_n\} = \operatorname{supp}\tau$.

LEMMA 6. *Let a measure σ satisfy the Kolmogorov–Szegö condition. Then*

$$\inf_{\tau} \min_{|z| \le r} |D_\tau(z)| > 0, \qquad \sup_{\tau} \max_{|z| \le r} |D_\tau(z)| < \infty,$$

where $0 < r < 1$, τ runs over the set of finitely supported measures such that $d_\tau = n_\tau - 1$.

Proof. It follows from

$$\int z^s \, d\sigma = \int z^s \, d\tau, \qquad 0 \le s \le n - 1$$

that

$$\int D_\tau(z) z^{-j} \, d\sigma = 0, \qquad 1 \le j \le n - 1. \tag{13}$$

Let

$$D_\tau(n) = z^n + \sum_{j=1}^{n} \alpha_j z^{n-j}.$$

Then

$$\int D_\tau(z) z^{-n} \, d\sigma = 1 + \sum_{j=1}^{n-1} \alpha_j \int z^{-j} \, d\sigma + \alpha_n \int z^{-n} \, d\sigma$$

and

$$\int D_\tau(z) \, d\sigma = \int z^n \, d\sigma + \sum_{j=1}^{n} \alpha_j \int z^{n-j} \, d\sigma.$$

In the corresponding relations for the measure τ the integrals containing D_τ vanish, i.e.,

$$0 = 1 + \sum_{j=1}^{n-1} \alpha_j \int z^{-j} \, d\tau + \alpha_n \int z^{-n} \, d\tau = 1 + \sum_{j=1}^{n-1} \alpha_j \int z^{-j} \, d\sigma + \alpha_n \int z^{-n} \, d\tau$$

and

$$\int z^n \, d\tau = - \sum_{j=1}^{n} \alpha_j \int z^{n-j} \, d\tau = - \sum_{j=1}^{n} \alpha_j \int z^{n-j} \, d\sigma.$$

Therefore,

$$\int D_\tau(z) z^{-n}\, d\sigma = \alpha_n \left(\int z^{-n}\, d\sigma + \overline{\sum_{j=1}^{n} \alpha_j \int z^{n-j}\, d\sigma} \right)$$

and, finally,

$$\int D_\tau(z) z^{-n}\, d\sigma = \alpha_n \overline{\int D_\tau(z)\, d\sigma}. \tag{14}$$

It is convenient to think of formula (14) in terms of the Hilbert space L^2_σ provided with the scalar product

$$(u, v) = \int u(z) \overline{v(z)}\, d\sigma.$$

Namely, $(D_\tau, z^n) = \alpha_n \overline{(D_\tau, 1)}$. But $D_\tau(z) = z^n + R(z) + \alpha_n$, where R is a polynomial of degree $\le n - 1$, $R(0) = 0$. Hence R is orthogonal to D_τ by (13). So

$$\|D_\tau\|^2 = (D_\tau, D_\tau) = (D_\tau, z^n) + \bar{\alpha}_n (D_\tau, 1).$$

As a result

$$\|D_\tau\|^2 = 2 \operatorname{Re} \bar{\alpha}_n (D_\tau, 1) = 2 \operatorname{Re}(D_\tau, z^n). \tag{15}$$

The formulas (15) show that $(D_\tau, 1) \neq 0$, $(D_\tau, z^n) \neq 0$. These inequalities and equalities (13) mean that D_τ is the so-called para-orthogonal polynomial of degree n with respect to the measure σ. In addition, D_τ is monic. Every such polynomial is of the form

$$D_\tau(z) = \frac{\Phi_n(z) + \theta_\tau \Phi_n^*(z)}{1 + \theta_\tau \overline{\Phi_n(0)}}, \tag{16}$$

where $|\theta_\tau| = 1$, Φ_n is the monic orthogonal polynomial ($\equiv$ Szegö polynomial), $\Phi_n^*(z) = z^n \overline{\Phi}_n(1/z)$, the bar means the conjugation of coefficients [4, Sect. 6].

Since all roots of $\Phi_n(z)$ lie inside the unit disk [3, Sect. 2.3] we have $|\Phi_n(0)| < 1$.

Obviously, for every fixed n and $n_\tau = n$

$$\sup_\tau \max_{|z| \le r} |D_\tau(z)| \le \frac{\max_{|z| \le r} \left(|\Phi_n(z)| + |\Phi_n^*(z)| \right)}{1 - |\Phi_n(0)|}.$$

On the other hand,

$$\inf_{\tau} \min_{|z|\le r} |D_\tau(z)| \ge \tfrac{1}{2} \min_{|\theta|=1} \min_{|z|\le r} |\Phi_n(z) + \theta\Phi_n^*(z)| > 0$$

since all roots of para-orthonormal polynomials lie on the unit circle [4, Theorem 6.2]. To finish the proof we note that the Kolmogorov–Szegö condition implies the asymptotic Szegö formula

$$\lim_{n\to\infty} \max_{|z|\le r} |\Phi_n(z)| = 0, \qquad \lim_{n\to\infty} \max_{|z|\le r} |\Phi_n^*(z) - f(z)| = 0,$$

where f is an analytic function in the disk $|z| < 1$ and $f(z)$ has no roots in it [3, Sect. 3.4]. ∎

It is obvious that Lemmas 5 and 6 imply Theorem 4.

Remark 7. There exists a different way to prove Lemma 6 using the Kolmogorov–Szegö criterion of the completeness for the system $\{z^s\}_1^\infty$ in L_σ^2. The author thanks the referee who suggested to use asymptotic Szegö formulas for a shorter proof.

Remark 8. The parameter θ_τ runs over the whole unit circle [4].

In conclusion we show that the Kolmogorov–Szegö condition is essential in Lemma 6.

Every difference equation

$$a_n y_{n+2} - (a_n + a_{n+1}z) y_{n+1} + a_{n+1} z(1 - |a_n|^2) y_n = 0 \qquad (17)$$

with $0 \ne |a_n| < 1$, $n = 0, 1, 2, \ldots$ defines a unique infinitely supported measure σ such that

$$\exp\left(\int \ln \sigma' d\omega \right) = \prod_{n=0}^{\infty} (1 - |a_n|^2)$$

and the solution of (17) under the initial conditions $y_0 = 1$, $y_1 = 1 - a_0 z$ is just the sequence of polynomials $\Phi_n^*(z)$ (see [7; 2, Chap. 8]. Therefore, the Kolmogorov–Szegö condition is violated if $\sum |a_n|^2 = \infty$.

EXAMPLE 9. Following [2] let $a_n = a$, $0 \ne |a| < 1$. Then (17) takes the form

$$y_{n+2} - (1 + z) y_{n+1} + z(1 - |a|^2) y_n = 0. \qquad (18)$$

Therefore,

$$\Phi_n^*(z) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} - (1 + a)z\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \tag{19}$$

where λ_1, λ_2 are roots of the characteristic equation

$$\lambda^2 - (1 + z)\lambda + z(1 - |a|^2) = 0.$$

Obviously, $\lambda_{1,2} = \frac{1}{2}(1 + z \pm \sqrt{\Delta(z)})$, where

$$\Delta(z) = 1 - 2(1 - 2|a|^2)z + z^2.$$

If x is real then $\Delta(x) > 0$. In this case one can choose the positive branch of $\sqrt{\Delta(x)}$. Then $\lambda_1(x) > |\lambda_2(x)|$ for $x > -1$ and it follows from (19) that, asymptotically,

$$\Phi_n^*(x) \approx \frac{\lambda_1(x) - (1 + a)x}{\sqrt{\Delta(x)}}\lambda_1^n(x) \tag{20}$$

as $n \to \infty$. For $x > 0$ we can insert x^{-1} instead of x. Since $x^2\Delta(x^{-1}) = \Delta(x)$ and $x\lambda_1(x^{-1}) = \lambda_1(x)$ we obtain from (20)

$$\Phi_n(x) \approx \frac{\lambda_1(x) - (1 + a)}{\sqrt{\Delta(x)}}\lambda_1^n(x). \tag{21}$$

Note also that by (18) all $\Phi_n^*(z)$ $(n \geq 1)$ have the same leading coefficient, namely $(-a)$.

Now we denote by τ_n the measure τ with $n_\tau = n$, $d_\tau = n - 1$ and

$$D_{\tau_n}(z) = \frac{\Phi_n(z) + \Phi_n^*(z)}{1 - a}$$

which corresponds to $\theta_\tau = 1$ in (16). From (20) and (21) we obtain

$$D_{\tau_n}(x) \approx \frac{2\lambda_1(x) - (1 + a)(1 + x)}{(1 - a)\sqrt{\Delta(x)}}\lambda_1^n(x).$$

The coefficient of this asymptotic relation is not zero if $x \neq 1$. Since $\lambda_1(x) > 1$ for $x > 0$ we obtain $D_{\tau_n}(x) \to \infty$ as $n \to \infty$ and $x > 0$, $x \neq 1$, moreover

$$\lim_{n \to \infty} \sqrt[n]{D_{\tau_n}(x)} = \lambda_1(x) > 1.$$

Therefore, the conclusion of Lemma 6 is not valid now.

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